Kit Fine's Ignorance of Ignorance

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- These sildes are based on Kit Fine, Ignorance of ignorance, 2017.

I am wiser than this man; it is likely that neither of us knows anything worthwhile, but he thinks he knows something when he does not, whereas when I do not know, neither do I think I know; so I am likely to be wiser than he to this small extent, that I do not think I know what I do not know. Socrates

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But there are also unknown unknowns –the ones we don't know we don't know. Rumsfeld

A Journey to Ignorance

1. What's the relation between ignorance and knowledge?

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2. Could we have a knowledge of ignorance?

Outline

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- What's the difference between ignorance and unknown?
- One is (first-order) ignorant whether p if one is both ignorant that p and ignorant not-p, and the notation is $Ip := \neg Kp \wedge \neg K \neg p$.
- One is second-order ignorant whether p if one is ignorant whether one is ignorant p, and the notation is IIp.

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Modal Logic Language

The basic modal logic language (BNF)

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \Box\varphi$$

where $p \in \mathbf{P}$, $\wedge, \rightarrow$ are defined as usual. $\Diamond$ is defined as $\neg\Box\neg$, and ∇p is defined as $(\Diamond p \wedge \Diamond\neg p)$. What's more, $\Box^2$ is short for$\Box\Box$; $\Diamond^2$ is for $\Diamond\Diamond$; ∇^2 for $\nabla\nabla$.

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Here are some intuitive ideas about the modalities.

- $\Box\varphi$: Someone knows φ
- $\nabla\varphi$: Someone is (first order) ignorant whether φ, i.e someone doesn't know φ and doesn't know $\neg\varphi$

Second-order Ignorance Implies First-order Ignorance

Lemma 1.

1. $\Box(\Box\varphi \vee \Box\neg\varphi)$ *is provably equivalent in* T *to* $\Box^2\varphi \vee \Box^2\neg\varphi$
2. $\Diamond(\Diamond\varphi \wedge \Diamond\neg\varphi)$ *is provably equivalent in* T *to* $\Diamond^2\varphi \wedge \Diamond^2\neg\varphi$

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Proof.

Just prove the first one, and the next can be easily proved by the duality.

$\Longleftarrow$

(1)	$\Box\varphi \to (\Box\varphi \vee \Box\neg\varphi)$	TH
(2)	$\Box(\Box\varphi \to (\Box\varphi \vee \Box\neg\varphi))$	**N**
(3)	$\Box\Box\varphi \to \Box(\Box\varphi \vee \Box\neg\varphi)$	(2), **K**, **MP**
(4)	$\Box\Box\neg\varphi \to \Box(\Box\varphi \vee \Box\neg\varphi)$	like the above
(5)	$(\Box\Box\varphi \vee \Box\Box\neg\varphi) \to \Box(\Box\varphi \vee \Box\neg\varphi)$	(3), (4), $\vee^+$

By the Deductive Theorem, $\Box^2\varphi \vee \Box^2\neg\varphi \vdash \Box(\Box\varphi \vee \Box\neg\varphi)$

Proof.

$\Longrightarrow$

(6)	$\Box(\Box\varphi \vee \Box\neg\varphi)$	**HY**
(7)	$\Box(\Box\varphi \vee \Box\neg\varphi) \to (\Box\varphi \vee \Box\neg\varphi)$	**T**
(8)	$\Box\varphi \vee \Box\neg\varphi$	(6), (7), **MP**
(9)	φ	**HY**
(10)	$\Box\neg\varphi$	**HY**
(11)	$\neg\varphi$	(10), **T**, **MP**
(12)	$\neg\Box\neg\varphi$	(9), (11), $\bot$
(13)	$\Box\varphi$	(8), (12), **MP**
(14)	$\Box(\varphi \wedge (\Box\varphi \vee \Box\neg\varphi)$	(6), (13), **RULE**
(15)	$\Box((\varphi \wedge \Box\varphi) \vee (\varphi \wedge \Box\neg\varphi))$	(14), **RULE**
(16)	$(\varphi \wedge \Box\varphi) \vee (\varphi \wedge \Box\neg\varphi)$	(15), **T**
(17)	$\varphi \wedge \Box\neg\varphi$	**HY**
(18)	$\bot$	

Proof.

(19)	$\neg(\varphi \wedge \Box\neg\varphi)$	(17), (18)
(20)	$\varphi \wedge \Box\varphi$	(16), (19)
(21)	$\Box(\varphi \wedge \Box\varphi)$	(20), **N**
(22)	$\Box\Box\varphi$	(21), **RULE**
(23)	$\neg\varphi$	**HY**
(24)	$\cdots$	
(25)	$\Box\Box\neg\varphi$	similarly
(26)	$\Box\Box\varphi \vee \Box\Box\neg\varphi$	(9), (23)

Now,we get the another side,i.e $\Box(\Box\varphi \vee \Box\neg\varphi) \vdash \Box^2\varphi \vee \Box^2\neg\varphi$ $\Box$

Lemma 2.

1. $\nabla^2\varphi$ *is provably equivalent in* T *to* $(\Diamond^2\varphi \wedge \Diamond\Box\neg\varphi) \vee (\Diamond^2\neg\varphi \wedge \Diamond\Box\varphi)$
2. $\neg\nabla^2\varphi$ *is provably equivalent in* T *to* $\Box^2\varphi \vee \Box^2\neg\varphi \vee \Box\nabla\varphi$
3. $\nabla^2\varphi$ *is provably equivalent in* $S4$ *to* $(\Diamond\varphi \wedge \Diamond\Box\neg\varphi) \vee (\Diamond\neg\varphi \wedge \Diamond\Box\varphi)$
4. $\neg\nabla^2\varphi$ *is provably equivalent in* $S4$ *to* $\Box\varphi \vee \Box\neg\varphi \vee \Box\nabla\varphi$

Lemma 2.

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4. $\neg\nabla^2\varphi$ *is provably equivalent in* $S4$ *to* $\Box\varphi \vee \Box\neg\varphi \vee \Box\nabla\varphi$

Proof.

1). $\nabla^2\varphi \dashv\vdash \Diamond\nabla\varphi \wedge \Diamond\neg\nabla\varphi$	the definition of ∇
$\dashv\vdash \Diamond(\Diamond\varphi \wedge \Diamond\neg\varphi) \wedge \Diamond\neg(\Diamond\varphi \wedge \Diamond\neg\varphi)$	the definition of ∇
$\dashv\vdash \Diamond^2\varphi \wedge \Diamond^2\neg\varphi \wedge \Diamond\neg(\Diamond\varphi \wedge \Diamond\neg\varphi)$	Lemma1.2
$\dashv\vdash \Diamond^2\varphi \wedge \Diamond^2\neg\varphi \wedge \Diamond(\Box\neg\varphi \vee \Box\varphi)$	Duality,
$\dashv\vdash \Diamond^2\varphi \wedge \Diamond^2\neg\varphi \wedge (\Diamond\Box\neg\varphi \vee \Diamond\Box\varphi)$	$\Diamond \vee$ *DIS*
$\dashv\vdash (\Diamond^2\varphi \wedge \Diamond^2\neg\varphi \wedge \Diamond\Box\neg\varphi) \vee (\Diamond^2\varphi \wedge \Diamond^2\neg\varphi \wedge \Diamond\Box\varphi)$	$\wedge$*DIS*
$\dashv\vdash (\Diamond^2\varphi \wedge \Diamond\Box\neg\varphi) \vee (\Diamond^2\neg\varphi \wedge \Diamond\Box\varphi)$	$\Box\Box\varphi \vdash \Box\Diamond\varphi$

Proof.

$$2).\neg\nabla^2\varphi \dashv\vdash \neg((\Diamond^2\varphi \wedge \Diamond\Box\neg\varphi) \vee (\Diamond^2\neg\varphi \vee \Diamond\Box\varphi)) \quad \text{Lemma2.1}$$

$$\dashv\vdash (\neg\Diamond^2\varphi \vee \neg\Diamond\Box\neg\varphi) \wedge (\neg\Diamond^2\neg\varphi \vee \neg\Diamond\Box\varphi) \quad \neg$$

$$\dashv\vdash (\Box^2\neg\varphi \vee \Box\Diamond\varphi) \wedge (\Box^2\varphi \vee \Box\Diamond\neg\varphi) \quad \text{Duality}$$

$$\dashv\vdash (\Box^2\neg\varphi \wedge \Box^2\varphi) \vee (\Box^2\neg\varphi \wedge \Box\Diamond\neg\varphi)$$

$$\vee (\Box\Diamond\varphi \wedge \Box^2\varphi) \vee (\Box\Diamond\varphi \wedge \Box\Diamond\neg\varphi) \quad \wedge DIS$$

$$\dashv\vdash \Box^2\varphi \vee \Box^2\neg\varphi \vee \Box\nabla\varphi \quad \bot,$$

$$\Box^2\neg\varphi \vdash \Box\Diamond\neg\varphi,$$

$$\Box^2\varphi \vdash \Box\Diamond\varphi,$$

$$(\Box\nabla\varphi)$$

The third and fourth lemmas can be easily proved in $S4$ by 1),2).and $\Diamond\varphi \to \Diamond\Diamond\varphi, \Box\varphi \to \Box\Box\varphi$. □

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Lemma 3.

$\nabla^2\varphi$ *provably implies* $\nabla\varphi$ *in* $S4$.

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Proof.

The proof can be turned into proof of $(\Diamond\varphi \wedge \Diamond\Box\neg\varphi) \vee (\Diamond\neg\varphi \wedge \Diamond\Box\varphi)$ because of Lemma 2.3. $\Diamond\Box\neg\varphi \vdash \Diamond\Diamond\neg\varphi \vdash \Diamond\neg\varphi \Longrightarrow (\Diamond\varphi \wedge \Diamond\Box\neg\varphi) \vdash \Diamond\varphi \wedge \Diamond\neg\varphi \Longrightarrow (\Diamond\varphi \wedge \Diamond\Box\neg\varphi) \vdash \nabla\varphi$; Similarly, we also have $(\Diamond\neg\varphi \wedge \Diamond\Box\varphi) \vdash \nabla\varphi$. So $\nabla^2\varphi \vdash \nabla\varphi$ $\Box$

The Impossibility of Knowing One Is Second-order Ignorant

Theorem 4.

The formula $\nabla^2\varphi$ *is unboxable in* $S4$.

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Proof.

We prove the theorem by contradiction.Suppose $\nabla^2\varphi$ is boxable in $S4$. By Lemma 3, $\Box\nabla^2\varphi \vdash \Box\nabla\varphi$;
$\Box\nabla^2\varphi \vdash \nabla^2\varphi \vdash (\Diamond\varphi \wedge \Diamond\Box\neg\varphi) \vee (\Diamond\neg\varphi \wedge \Diamond\Box\varphi)$(Lemma 2.3), and then $\Box\nabla^2 \vdash \Diamond\Box\neg\varphi \vee \Diamond\Box\varphi$.
So $\Box\nabla^2\varphi \vdash \Box\nabla\varphi \wedge (\Diamond\Box\neg\varphi \vee \Diamond\Box\varphi) \vdash (\Diamond\Box\neg\varphi \wedge \Box\nabla\varphi) \vee (\Diamond\Box\varphi \wedge \Box\nabla\varphi)$
$(\Diamond\Box\neg\varphi \wedge \Box\nabla\varphi) \vdash \Diamond(\Box\neg\varphi \wedge \nabla\varphi) \vdash \bot$(*By* $\vdash_{\mathbf{K}} (\Diamond\varphi \wedge \Box\psi) \rightarrow \Diamond(\varphi \wedge \psi), \nabla_{df}$);
Similarly, we can get $(\Diamond\Box\varphi \wedge \Box\nabla\varphi) \vdash \Diamond(\Box\varphi \wedge \nabla\varphi) \vdash \bot$;
Finally, $\Box\nabla^2\varphi \vdash \bot$, and $\nabla^2\varphi$ is unboxable in $S4$. □

The Equivalence of High-order Ignorance and Second-order Ignorance

Corollary 5.

For $m, n > 1, \nabla^m\varphi \dashv\vdash \nabla^n\varphi$ in $S4$.

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Proof.

$\nabla^3\varphi \vdash \nabla^2\varphi$(Lemma 3($\nabla^2\varphi/\nabla\varphi$));
$\nabla^2\varphi \vdash \Diamond\nabla^2\varphi$(**RP**)
$\nabla^2\varphi \vdash \Diamond\nabla^2\varphi \wedge \Diamond\neg\nabla^2\varphi$(Theorem 4)
$\nabla^2\varphi \vdash \nabla^3\varphi$;
$\nabla^2\varphi \dashv\vdash \nabla^3\varphi*$;
$\nabla^k\varphi \dashv\vdash \nabla^{k+1}\varphi(k > 1)$(Substitution of *) □

The Distributions of Truth-value about Ignorance

- A contingency profile α is an infinite sequence $\alpha_1\alpha_2\alpha_3\ldots$ of T's and F's.

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- A contingency profile α is an infinite sequence $\alpha_1\alpha_2\alpha_3\ldots$ of T's and F's.
- A set corresponding to a contingency profile α is $\nabla_\alpha = \{\varphi_1, \varphi_2, \varphi_3 \ldots\}$, where φ_k is $\nabla^k p$ when α_k is T and $\neg\nabla^k p$ when α_k is F. It is called the corresponding set of a contingency profile α.

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- A contingency profile α is said to be consistent in the system S if the corresponding set ∇_α is consistent in S.

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A contingency profile α is consistent in S4 iff it is of the form $FFF\dots$ or $TFF\dots$ or $TTT\dots$

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Proof.

By Corollary 5, a consistent contingency profile must be of the form $\alpha_1 FF\cdots$ or $\alpha_1 TT\cdots$ It is readily shown that if $\alpha_1 = F$ then each$\alpha_k = F$.

Model 1(the contingency of FFF: $\circ\ p$

Model 2(the contingency of TFF: $\circ \longleftrightarrow \circ\ p$

Model 3(the contingency of TTT: $\circ \longrightarrow \circ\ p$ □

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Model 3(the contingency of TTT: $\circ \longrightarrow \circ\ p$ □

Not ignorance, but ignorance of ignoance, is the death of knowledge. Alfred North Whitehead

Subject Matter

We have talked much about the (high-order) ignorance of a fact. And a very natural question would be asked: what would it be for some subject matther?

Some General Results of Ignorance about Subject Matter

Theorem 7.

For $n > 0$, the formulas $\nabla(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$, $\nabla(\nabla p_1 \vee \nabla p_2 \vee \ldots \vee \nabla p_n)$, $\nabla^2 p_1 \wedge \nabla^2 p_2 \wedge \ldots, \wedge \nabla^2 p_n$ and $\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_n$ are unboxable in $S4$.

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For $n > 0$, the formulas $\nabla(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$, $\nabla(\nabla p_1 \vee \nabla p_2 \vee \ldots \vee \nabla p_n)$, $\nabla^2 p_1 \wedge \nabla^2 p_2 \wedge \ldots, \wedge \nabla^2 p_n$ and $\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_n$ are unboxable in $S4$.

Proof.

Let us deal with the formula $\nabla(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$(the proof of the second formula is similar).

$\nabla(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$

$\vdash \Diamond(\Diamond p_1 \wedge \Diamond\neg p_1 \wedge \Diamond p_2 \wedge \Diamond\neg p_2 \wedge \ldots \wedge \Diamond p_n \wedge \Diamond\neg p_n)$ by def. of ∇, T

$\vdash \Diamond\Diamond p_1 \wedge \Diamond\Diamond\neg p_1 \wedge \Diamond\Diamond p_2 \wedge \Diamond\Diamond\neg p_2 \wedge \ldots \wedge \Diamond\Diamond p_n \wedge \Diamond\Diamond\neg p_n$ by theorem in T

$\vdash \Diamond p_1 \wedge \Diamond\neg p_1 \wedge \Diamond p_2 \wedge \Diamond\neg p_2 \wedge \ldots \wedge \Diamond p_n \wedge \Diamond\neg p_n$ by$\Diamond\Diamond \vdash \Diamond$

Hence:

$\Box\nabla(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n) \vdash$

$\Box(\Diamond p_1 \wedge \Diamond\neg p_1 \wedge \Diamond p_2 \wedge \Diamond\neg p_2 \wedge \ldots \wedge \Diamond p_n \wedge \Diamond\neg p_n)$**

Also: $\Box\nabla(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$

Proof (cont.)

$\vdash \nabla((\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$
$\vdash \Diamond\neg(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$
$\vdash \Diamond(\neg\nabla p_1 \vee \neg\nabla p_2 \vee \ldots \vee \neg\nabla p_n)$
$\vdash \Diamond(\Box p_1 \vee \Box\neg p_1 \vee \Box p_2 \vee \Box\neg p_2 \vee \ldots \vee \Box p_n \vee \Box\neg p_n)$ by def. of ∇
But then , given ** above, it follows that:
$\Box\nabla(\nabla p_1 \wedge \nabla p_2 \wedge \ldots \wedge \nabla p_n)$
$\vdash \Diamond((\Diamond p_1 \wedge \Diamond\neg p_1 \wedge \Diamond p_2 \wedge \Diamond\neg p_2 \wedge \ldots \wedge \Diamond p_n \wedge \Diamond\neg p_n) \wedge (\Box p_1 \vee \Box\neg p_1 \vee \Box p_2 \vee \Box\neg p_2 \vee \ldots \vee \Box p_n \vee \Box\neg p_n))$ by $\ldots, \Box \vdash \Diamond$, def. of ∇
$\vdash \bot$ by the duality.
The proof for the formula $\nabla^2 p_1 \wedge \nabla^2 p_2 \wedge \ldots \wedge \nabla^2 p_n$ is straightforward since if $\nabla^2 p_1 \wedge \nabla^2 p_2 \wedge \ldots \wedge \nabla^2 p_n$ were boxable then $\nabla^2 p_1$ would also be boxable, which is contradictory to Theorem 4.
The final case is the formula $\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_n$. We prove $\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_n)$ inconsistent in $S4$ by induction on n. The case $n = 1$ is Theorem 4. Consider the case $n = k + 1$.What we need to prove is
$\neg\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_k) \Rightarrow \neg\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_{k+1})$.

Proof (cont.)

We prove it by contradiction. So suppose $\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_{k+1})$, then $\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_{k+1}$ and so $(\Diamond p_1 \wedge \Diamond\Box\neg p_1) \vee (\Diamond\neg p_1 \wedge \Diamond\Box p_1) \vee (\Diamond p_2 \wedge \Diamond\Box\neg p_2) \vee (\Diamond\neg p_2 \wedge \Diamond\Box p_2) \vee \ldots \vee (\Diamond p_{k+1} \wedge \Diamond\Box\neg p_{k+1}) \vee (\Diamond\neg p_{k+1} \wedge \Diamond\Box p_{k+1})$(Lemma 2.3). Now we suppose $(\Diamond p_{k+1} \wedge \Diamond\Box\neg p_{k+1})$, and so, in particular, $\Diamond\Box\neg p_{k+1}$(we will have $\Diamond\Box\Box\neg p_{k+1}$. Given $\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_{k+1})$, $\Diamond(\Box\Box\neg p_{k+1} \wedge \Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_{k+1}))$ and so $\Diamond\Box(\Box\neg p_{k+1} \wedge (\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_{k+1}))$. But $\Box\neg p_{k+1}$ is inconsistent with $\nabla^2 p_{k+1}$ and so $\Diamond\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_k))$,which this is inconsistent by the inductive hypothesis.$[\neg\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_k) \wedge \Diamond\Box(\nabla^2 p_1 \vee \nabla^2 p_2 \vee \ldots \vee \nabla^2 p_k)) \vdash \bot]$ $\Box$

Three powerful formulas

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Any p-formula $\Box\psi$*, for* ψ *non-modal, is equivalent in* T *to:*
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Proof.

Any non-modal p-formula ψ is equivalent to $\top, \bot, p$, or $\neg p$; and from this it is readily shown that any p-formula $\Box\psi$, for ψ non-modal, is equivalent to $\top, \bot, \Box p$, or $\Box\neg p$ □

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The formula φ is complete for a class of formulas Γ in the modal system S if for any formula ψ of Γ either $\varphi \vdash_S \psi$ or $\varphi \vdash_S \neg\psi$

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In S4, each of the formulas $\Box p, \Box\neg p$ and $\Box\nabla p$ is complete for the class of modalized p-formulas.

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In S4, each of the formulas $\Box p, \Box\neg p$ and $\Box\nabla p$ is complete for the class of modalized p-formulas.

Proof.

For the class of all p-formulas of the form $\Box\psi$, ψ is non-modal. It is easy to show that each of the formulas $\Box p, \Box\neg p,$ and $\Box\nabla p$ is complete for the class of formulas $\{\top, \bot, \Box p, \Box\neg p\}$ and hence, by the previous lemma, it is complete for $\Box\psi$.
It is easy to show that completeness is preserved under negation and conjunction, i.e. that if φ is complete for Γ then it is complete for $\{\neg\chi : \chi \in \Gamma\}$ and for $\{\chi \wedge \chi' : \chi, \chi' \in \Gamma\}$. It therefore suffices to show that completeness is preserved under necessitation, i.e. that if φ is complete for Γ then it is complete for $\{\Box\chi : \chi \in \Gamma\}$.

Proof (cont.)

For suppose $\Box\psi$ is complete for the class of formulas Γ. Take now any formula of the form $\Box\psi$. For suppose $\Box\psi$ is complete for the class of formulas Γ. Take now any formula of the form $\Box\chi$ for $\chi \in \Gamma$. Then either $\Box\psi \vdash \chi$ or $\Box\psi \vdash \neg\chi$. In the former case,$\Box\psi \vdash \Box\Box\psi \vdash \Box\chi$ and similarly for the latter, $\Box\psi \vdash \Box\Box\psi \vdash \Box\neg\chi \vdash \neg\Box\chi$.(**D**) $\Box$

More about the Impossibility

Theorem 10.

In the system S4, the modalized p-formula φ is unboxable iff it implies $\nabla\nabla p$.

More about the Impossibility

Theorem 10.

In the system S4, the modalized p-formula φ is unboxable iff it implies $\nabla\nabla p$.

Proof.

The right to left direction follows from Theorem 4. For the other direction, suppose that φ does not imply $\nabla\nabla p$. Then φ is consistent with $\neg\nabla\nabla p$ and hence, by Lemma 2.4, consistent with $\Box p \vee \Box\neg p \vee \Box\nabla p$. But then φ is consistent with $\Box p$ or $\Box\neg p$ or $\Box\nabla p$. So by Lemma 9, one of these three formulas of the form $\Box\psi$ implies φ. But then $\Box\psi$ implies $\Box\varphi$ and, since $\Box\psi$ is consistent then so is $\Box\varphi$, which means $\Box\varphi$ is consistent with φ.It is contradictory to which the modalized p-formula φ is unboxable. □

More about the Impossibility

Theorem 11.

In S4, there is no weakest unboxable p-formula,i.e. no unboxable p-formula implied by all unboxable formulas.

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Proof.

$p \wedge \Diamond\neg p$ and $\neg p \wedge \Diamond p$ are both unboxable. Now if there were a weakest unboxable formula φ, it would be implied by both $p \wedge \Diamond\neg p$ and $\neg p \wedge \Diamond p$ and hence implied by their disjunction $(p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond p)$, which is equivalent in T to ∇p. But $\Box\nabla p$ is consistent in $S4$ and hence so is $\Box\varphi$ and φ would be boxable after all. $\Box$

Outline

Modal Degree and Some Other Notations

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$$\begin{aligned} md(p) &= 0 \\ md(\neg\varphi) &= md(\varphi) \\ md(\varphi \vee \psi) &= max(md(\varphi), md(\psi)) \\ md(\Box\varphi) &= md(\varphi) + 1 \end{aligned}$$

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- Given a binary relation R on a set W, we take $wR^n v$ to hold, for $w, v \in W$ and $n \geq 0$, if there is a sequence of elements $w_0, w_1, \ldots, w_n$ for $w_0 = w, w_n = v$ and $w_k R w_{k+1}$ for $k = 0, 1, \ldots, n-1$

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- Given a Kripke model $\mathbf{M} = (W, R, V, w)$, we let $\mathbf{M}^n$, for $n \geq 0$, be the model (W^n, R^n, V^n, w), where $W^n = \{v \in W : wR^k v \text{ for } k \leq n\}$ and R^n and V^n are the restriction of the R and V to W^n.

A Finite Submodels

Lemma 12.

Given any formula φ *of modal degree* $\leq n$ *and any model* $\mathbf{M} = (W, R, V, w)$*:* φ *is true at* $\mathbf{M}$ *iff* φ *is true at* $\mathbf{M}^n$*.*

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Proof.

when $n = 0$, φ does not contains $\Box$.So it is clear that φ is true at $\mathbf{M}$ *iff* ϕ is true at $\mathbf{M^0}$

suppose $n = m + 1$,φ is true at $\mathbf{M}$. It means $\Box\psi$ is true at $\mathbf{M}$. And we suppose wRw_1, then ψ is true at $\mathbf{M}' = (W, R, V, w_1)$. By induction hypothesis, ψ is true at $\mathbf{M}'^m = (W^m, R^m, V^m, w_1)$. Because of wRw_1, $W^{m+1} = \{v \in W : wR^{k+1}v \text{ for } k \leq m\}$, which means $\Box\psi$ is true at M^{m+1}.

The other direction can be easily proved like the above. $\Box$

Strip Model and It's Extension

- By a strip model of order $n, n \geq 0$, is meant a Kripke-model of the form $\mathbf{M} = (W, R, V, 0)$, where:
  1. $W = \{0, 1, 2, \ldots, n\}$
  2. $R = \{\langle k, k+1 \rangle : k = 0, 1, 2, \ldots, n-1\} \cup \{\langle k, k \rangle : k \in W\}$
  3. V is represented by a sequence $p_0 p_1 \ldots p_n$ of length $n + 1$, where p_k is p should p be true at k and p_k is $\bar{p}$ should p be false at k.

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- The strip model $\mathbf{M}^+ = (W^+, R^+, V^+, 0)$ of order $n+1$ extends the strip model $\mathbf{M} = (W, R, V, 0)$ of order n if V is the restriction of V^+ to W, i.e. if the truth-value of p at the worlds $0, 1, 2, \ldots, n$ are the same in each model.

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- The strip model $\mathbf{M}^+ = (W^+, R^+, V^+, 0)$ of order $n+1$ extends the strip model $\mathbf{M} = (W, R, V, 0)$ of order n if V is the restriction of V^+ to W, i.e. if the truth-value of p at the worlds $0, 1, 2, \ldots, n$ are the same in each model.
- $\pm\varphi$ is used for φ or $\neg\varphi$.

Strip Model and It's Extension

Lemma 13.

For each strip model $\mathbf{M} = (W, R, V, 0)$ *of the order* n *and for each formula* $\pm\nabla^{n+1}p, n \geq 0$*, there is an extension* $\mathbf{M}^+ = (W^+, R^+, V^+, 0)$ *of* $\mathbf{M} = (W, R, V, 0)$ *at which* $\pm\nabla^{n+1}p$ *is true.*

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Proof.

By induction on n. For each n,there are two cases to consider: when $\pm\nabla^{n+1}p = \nabla^{n+1}p$ and when $\pm\nabla^{n+1}p = \neg\nabla^{n+1}p$.

n=0. The given model $\mathbf{M} = (W, R, V, 0)$ is of the form $(\{0\}, R, V, 0)$ and the associated sequence is either p or $\bar{p}$.

Suppose first that $\pm\nabla^{1}p = \nabla p$.If the associated sequence is p, we may let the sequence associated with $\mathbf{M}^{+}$ be $p\bar{p}$; and if the associated sequence is $\bar{p}$, we may let the sequence associated with $\mathbf{M}^{+}$ be $\bar{p}p$.

Now suppose that $\pm\nabla^{1}p = \neg\nabla p$. If the associated sequence is p, we may let the sequence associated with $\mathbf{M}^{+}$ be pp; and if the associated sequence is $\bar{p}$, we may let the sequence associated with $\mathbf{M}^{+}$ be $\bar{p}\bar{p}$.

Proof (cont.)

$n = k+1$ The given model $\mathbf{M} = (W, R, V, 0)$ is of the form $(\{0, 1, \dots, k+1\}, R, V, 0)$
Suppose first that $\pm\nabla^{k+1}p = \Diamond\nabla^{k+1}p = \Diamond\nabla^{k}p \wedge \Diamond\neg\nabla^{k}p$. There are two subcases.(a) $\nabla^{k}p$ is true at $\mathbf{M}$. Consider the model $\mathbf{N} = (W, R, V, 1)$. By induction hypothesis, there is an extension $\mathbf{N}^{+}$ of $\mathbf{N}^{k}$(which no longer contains the point 0) at which $\neg\nabla^{k}p$ is true. Let $\mathbf{M}^{+}$ be the corresponding extension of $\mathbf{M}$ (in which the point 0 is restored). Then $\nabla^{k}p$ will be true at 0 in $\mathbf{M}^{+}$ and $\nabla^{k}p$ false at 1 in $\mathbf{M}^{+}$; and so $\nabla^{k+1}p$ will be true at $\mathbf{M}^{+}$.(b) $\nabla^{k}p$ is similar to that for (a) but with the role of $\nabla^{k}p$ and $\neg\nabla^{k}p$ reversed.
Now suppose $\pm\nabla^{k+1}p = \neg\nabla^{k+1}p$. Again, there are two subcases.(a) $\nabla^{k}p$ is true at $\mathbf{M}$. By IH, there is an extension $\mathbf{N}^{+}$ of $\mathbf{N}^{k}$ at which $\nabla^{k}p$ is true(with $\mathbf{N}$ as before). Let $\mathbf{M}^{+}$ be the corresponding extension of $\mathbf{M}$. Then $\nabla^{k}p$ will be true at 0 in $\mathbf{M}^{+}$ and $\nabla^{k}p$ true at 1 in $\mathbf{M}^{+}$; and so $\Box\nabla^{k}p$ and hence $\neg\nabla^{k+1}p$ will be true at $\mathbf{M}^{+}$.(b) $\nabla^{k}p$ is false at $\mathbf{M}$. By IH,there is an extension $\mathbf{N}^{+}$ of $\mathbf{N}^{k}$ at which $\neg\nabla^{k}p$ is true... and so $\Box\neg\nabla^{k}p$ and hence $\neg\nabla^{k+1}p$ will be true at $\mathbf{M}^{+}$. □

Absolute Knowledge of n-th Order Ignorance

Theorem 14.

Each set of formulas $\{\pm\nabla^1 p, \pm\nabla^2 p, \pm\nabla^3 p, \ldots\}$ *is consistent in the system* T.

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Proof.

By Lemma 14, there will be s strip model $\mathbf{M}_1$ (of order 1) at which $\pm\nabla^1 p$, an extension $\mathbf{M}_2$ of $\mathbf{M}_1$ at which $\pm\nabla^1 p$ and $\pm\nabla^2 p$ are true; and so on. The union $\mathbf{M}$ of the models $\mathbf{M}_1, \mathbf{M}_2, \ldots$ will be a model for $\{\pm\nabla^1 p, \pm\nabla^2 p, \pm\nabla^3 p, \ldots\}$ since each of $\mathbf{M}_1, \mathbf{M}_2, \ldots$ are generated submodels of $\mathbf{M}$ and hence will agree with $\mathbf{M}$. □

Absolute Knowledge of *n*-th Order Ignorance

- For each formula φ, let $\Box^\infty\varphi$ be the set of formulas $\{\varphi, \Box\varphi, \Box^2\varphi, \Box^3\varphi, \ldots\}$.

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For each $n = 1, 2, \ldots, \Box^{\infty}\nabla^{n}p$ *is consistent in* T.

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Corollary 15.

For each $n = 1, 2, \ldots, \Box^\infty\nabla^n p$ is consistent in T.

Proof.

By Theorem 15, $\{\nabla^n p, \neg\nabla^{n+1}p, \neg\nabla^{n+2}p, \ldots\}$ is consistent in T. $\neg\nabla^{n+1}p \vdash \Box\nabla^n p \vee \Box\neg\nabla^n p$. Because of $\nabla^n p$, it is clear that $\Box\nabla^n p$ is consistent in T. And by $\Box \vdash \Box\Box$, we also have $\Box\Box\nabla^n p, \ldots,$ is a consistent in T. So $\Box^\infty\nabla^n p$ is consistent in T. □

Outline

Some Comments

1. A summary

Some Comments

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2. Semantics

Some Comments

1. A summary
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3. Text structure

Some Comments

1. A summary
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3. Text structure
4. About my research
   - nested modalities

Some Comments

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   - nested modalities
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Thanks!